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A GRAPHICAL CONSTRUCTION FOR THE DEFLECTION OF STRUCTURAL MEMBERS OF VARIABLE MOMENT OF INERTIA

by Anthony Hoadley

STRUCTURAL DIVISION

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A GRAPHICAL CONSTRUCTION FOR THE DEFLECTION OF STRUCTURAL MEMBERS OF VARIABLE MOMENT OF INERTIA

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SYNOPSIS

A method is developed by which the deflections of all points on a curved beam of variable moment of inertia can be found from the bending moment curve for the loading under consideration. The method is analagous to the Williot-Mohr construction for finding the deflection of trusses. In the truss relative joint deflections are found due to change in length of truss members. Here the relative motions of points along the neutral axis are found from the bending stress in each segment of the beam.

The resulting deflection diagram gives both x and y components for all points along the structural member. The construction of the deflection diagram can be extended to include the effect of direct stress if that is large enough to be of importance.

DEVELOPMENT OF GEOMETRICAL RELATIONS

The problem of finding the deflections in a structure or machine may be of direct interest as in determining the camber needed to overcome dead and live load deflections in structures or in checking clearances in machinery. Indirectly, deflection diagrams are very useful tools in the construction of influence lines. This use of the deflection diagram may be its most important use.

The method to be described makes it possible to construct the deflected position of a curved or irregularly shaped structural member of varying moment of inertia providing that the moment diagram can be obtained for the loading considered. The method is a semi-graphical one that, in a single diagram, gives the absolute deflection of each point on the structure under the chosen loading. It is assumed that the deflections are not large enough to materially change the geometry of the structure. Basically the method consists in finding the relative deflections of points at the ends of each segment of a curved beam, graphically determining the deflections of these points from an assumed tangent and then rotating the deflected beam until the points of support reach their known relative positions.

The geometrical relationships involved in finding the motion of segments of a curved beam relative to the fixed end of the beam at point O are shown in Fig. 1. The light line represents segments 1 and 2 of the unloaded beam. The heavy line represents the final position of these segments due to the deflection of the beam. If segment 2 were perfectly rigid its position due to the deflection of segment 1 would be as shown by the light dash lines.

Due to the applied loads the left face of segment 1 undergoes a deflection a-a' and has an angular displacement of $\Delta\theta_1$. The point b in the left face of segment 2, which is considered to be rigid, is moved to point b' due to the rotation of segment 1. Due to the deflection a-a' of point a, point b is moved an equal distance b'-b" in the direction of a-a'. Then due to the bending of segment 2 point b is deflected from b" to its final position b"'. It is here assumed that the deflections are small enough so that the line b" b"' is perpendicular to the chord ab. The deflection of point b relative to point a is thus equal to the length of segment 2 times the angular rotation at point a plus the deflection of b from the tangent at a.

For all practical purposes, due to the relative magnitude of the angular changes and of the displacements in comparisons with the length of the segment, it is assumed that the deflection of a segment takes place in a direction perpendicular to the chord of that segment. The close agreement between analytical and graphical solutions reported later in this paper would seem to substantiate the validity of the assumptions made.

It can now be seen that the displacements of the ends of the various segments can be found graphically by plotting to scale successively the deflection a-a' at right angles to chord o-a of segment 1, next the sum of the displacements b-b' and b"-b"' at right angles to chord ab of segment 2 and continuing in similar fashions for subsequent segments.

The change in slope in a differential length of the segment ds is $\int \frac{m\,ds}{EI}$

The total angular change $\Delta\theta_1$ in length ΔS_1 will then be $= \int \frac{m\,ds}{EI} = \frac{M\Delta S_1}{EI}$

where ΔS_1 should be chosen so that the above equality is true within the desired degree of accuracy. Special care must be used in laying out the segments near a sharp peak in the moment curve such as that found at the support of a continuous beam. The displacement a-a' of Fig. 1 will be $\int \frac{1 M\,ds}{EI} = \Delta\frac{S_1}{2} \times \frac{(M\Delta S_1)}{EI}$ and will be in a direction nearly perpendicular to to the chord O-a of segment 1. The displacement b-b' will be $\Delta S_2 \times \Delta\theta_1 = \Delta S_2\left(\frac{M\Delta S_1}{EI}\right)$ and will be perpendicular to chord ab of segment 2. For a comment on the differences between points b and b' see the Appendix.

In general terms the deflection of point n with respect to point n-1 is given by Equation I or its equivalent Equation II.

$$\frac{\Delta Sn}{2}\left(\frac{Mn\Delta Sn}{EI}\right) + \Delta Sn \sum_{o}^{n-1} \frac{M\Delta S}{EI} \qquad \text{Equ. I}$$

$$\frac{\Delta Sn}{2}(\Delta\theta n) + Sn \sum_{o}^{n-1} \Delta\theta \qquad \text{Equ. II}$$

Mn = Average bending moment in segment n
m = Bending moment at any point along the beam
ΔS = Length of segment measured along neutral surface.*
I = Average moment of inertia of the segment.*
*See Appendix.

Circular Rib With Constant I. The deflected position of the circular rib shown in Fig. 2 will be found to illustrate the application of this graphical method. The rib shown has a constant radius of curvature of 12.75 ft. and constant moment of inertia. It is fixed at O and carries a moment of 100 foot kips at point 10. The general procedure would be the same for a beam of variable moment of inertia and radius of curvature. Any type of loading could be applied. The conditions used in this problem simplify the analytical solutions used to check the graphical results.

The relative deflections of the arch rib segments must first be computed. This is done in Table I. As both moment of inertia and bending moment are constant the angular change $\Delta\theta$ for each segment is $\frac{100 \times 2}{EI}$. The total deflection of segment n (Col. 6) with respect to segment (n-1) is equal to the deflection due to the bending of segment n (Col. 3) plus the deflection due to the rotation of segment n (Col. 5). The value of Δ_S has been taken as 2 ft. for each segment.

From point O' lay off the distance 200 in the direction of the radius to the center of segment 1, establishing point 1' of Fig. 2B. From 1' draw 1'-2' 600 units long parallel to the radius to the center of segment 2. In a similar fashion locate all of the points to 10'. 0'-1' represents the motion of 1' with respect to the fixed point 0. 1'-2' gives the motion of point 2 relative to point 1 so that 0'-2' represents the absolute motion of point 2. The absolute motion of any point on the beam will be given by a line drawn from 0' to the corresponding point in Fig. 2b.

The deflected position of the loaded circular beam is now found by drawing to scale lines 1-1", 2-2" etc. (Fig. 2A) parallel to rays 0'-1', 0'-2' etc. of Fig. 2B. The curve connecting points 0, 1", 2" -10" was found to be circular as would be expected in the case of a beam loaded with a bending moment and carrying no shear. The horizontal and vertical components of the deflection of point 10 agree with those found by an analytical solution with an error of less than 1%.

REINFORCED CONCRETE ARCH OF 123 FT. SPAN CARRYING A SINGLE LOAD

The deflected position of a fixed-ended arch of 123 ft. span carrying a 10 kip load near the quarter-point has been found and is shown in Fig. 4. The computations are shown in Table II. The sum of the angular changes computed in Col. 4 is not zero as it should be. The error was distributed proportionately to the size of the angle change in each segment and the adjusted angle entered in Col. 6. Data on the arch axis were taken from "Theory and Practice of Reinforced Concrete" by Clarence W. Dunham, Member ASCE. Under the loading used Fig. 4 shows that the middle half of the arch has an appreciable shift to the left. The vertical component of the deflection at the left quarter point was found to be 0.230" as compared with the 0.236" given by Prof. Dunham's analytical solution. This deflection curve is also the influence line for deflection at panel point 6.

Effect of Direct Stress: The effect of direct stress on the deflection of the arch rib can be shown in the following way. Draw a line from point 1 of the Relative Deflection Diagram so that it is parallel to the thrust acting on segment 1 of the arch. The length of this line should equal the shortening of segment 1 due to direct stress and will extend downward to the left from

point 1 of the Relative Deflection Diagram. This new point gives the deflection of the right end of segment 1 due to both direct stress and bending. The effect of direct stress in the other segments is found in a similar fashion. The effect of direct stress in the arch used above was negligible for a 10,000 lb. load at panel point 6'.

DEFLECTION OF CURVED BEAMS WITH HINGED ENDS

In the case of structures which do not have a fixed point or a point at which the slope of the tangent to the neutral axis is known it becomes necessary to find the deflections of panel points with respect to an assumed tangent and then to rotate the structure until the points of support have the correct relative elevation.

Rigid Frame Bridge - Dead Load Deflections. The dead load deflection curve for an unsymmetrical rigid-frame bridge with hinged ends will be found to illustrate the solution of a problem of this type. Data on the bridge and its dead load moments are taken from "The Rigid-Frame Bridge" by Arthur G. Hayden and Maurice Barron, Members ASCE. The deflections of panel points relative to point 11 1/2 were computed from the dead load moments and elastic properties of the bridge on a tabular form similar to that shown in Table II. The relative deflection diagram was then drawn as shown by the light lines and primed points of Fig. 3B. The light solid line in Fig. 3A shows the position of the bridge relative to a tangent to the neutral axis at panel point 11 1/2. This shows that with point 11 1/2 fixed the left end of the bridge has moved to the right and slightly below its original position. At the same time the right end of the bridge has moved to the right and a considerable distance above its original position. It is now necessary to rotate the bridge about its left end until the right end is the original distance of 6.90 ft. above the left end. The motion of any panel point caused by this rotation will be proportional to its distance from the left hinge and will be perpendicular to the line connecting these two points. In Fig. 3B the vertical distance between O and R represents, to the scale of the drawing, the distance that the right hand end of the bridge must be dropped to bring the ends of the bridge to their correct elevations. Starting from O draw a rigid-frame whose span is O-R and rotated 90° with respect to the real bridge. The distance from a panel point on this correction diagram to O will then represent the motion of the corresponding point of the rigid-frame when the rotation takes place. For panel point 17 this is shown as 17"-O. The motion of point 17 relative to point O is O-17'. Its absolute motion is the resultant of these motions or the distances 17"-17' in Fig. 3B. The deflections in Fig. 3a are shown to just half the scale used in drawing deflections in Fig. 3b. The deflections shown are for the face of each segment which is farthest from point 11 1/2.

Deflection Curve for Continuous Beam. The construction of the deflection curve for a beam 20 ft. long which is supported at its left end and midpoint and which carries a one pound load acting vertically upwards at its right end is shown in Table III and Fig. 5. The beam has a constant moment of inertia and Δ_S is taken as 2 ft. Substituting these values in Equ. I and multiplying by EI gives

$$EI \times \text{Relative deflection} = 2M_n + \sum_{o}^{n-1} 4M$$

of segments n and (n-1)

where M is the moment at the center of a segment due to the one pound load. The relative deflections shown in Col. 4 of Table III (which are EI x actual relative deflection) are plotted from point 10' of Fig. 5b in a direction perpendicular to the tangent to the beam at the center of each segment. This results in the straight line 10' - 20'. The deflections 10' -12', 10' - 14' etc. are then plotted at the appropriate points on the beam giving the dotted deflection curve of Fig. 5c which is the deflection curve relative to a tangent at 10. This shows 0' to be the distance 10' -0' above 0. Actually the left end of the beam must rest on the point of support 0. This condition will be attained when the dotted deflection curve is rotated until 0' drops to 0. This can be done by drawing a Mohr correction diagram as shown in Fig. 5b. The diagram has been drawn at right angles to the original beam with the length of one span euqal to the distance 0' must drop or 0' -10' on the relative deflection diagram. The distance from a point on the Mohr diagram to the corresponding point on the relative deflection diagram will give the absolute deflection of that point on the beam. These scaled distances were then recorded in Col. 5 of Table III and then used to plot the final deflection curve shown in Fig. 5c. The actual deflections would be the value of Col. 5 divided by EI. This method of finding deflection curves for beams can be used for haunched or reinforced beams by using the average I for each segment in carrying out the summation used in Equation I. The arithmetic becomes a little more difficult but that is all.

Influence Line for Continuous Beam. Muller-Breslaw, D. B. Steinman, Member A.S.C.E., and others have pointed out that the deflection curve due to a load acting in place of the reaction of an indeterminate beam is proportional to the influence line for the redundant reaction. The influence ordinates for a vertical reaction at point 20 were obtained by dividing the scaled deflections of Col. 5 by 663.

APPENDIX

Chord vs Arc. For computing the rotational effect the correct distance to use is the chord (ab of Fig. 1). For bending of the segment ΔS as measured along the arc should be used. When ΔS for a curved beam is so chosen that the angle subtended by the segment is 20^{o} the arc will be 1/2% longer than the segment. For a subtended angle of 28^{o} the differences in length will be 1%. Segment lengths can readily be chosen so that the difference in length of chord and arc is less than the accuracy obtainable in the drawing.

Direction of Rotational Deflection. The following study shows that, even in an extreme case of elastic deflection, the tangent b-b' of Fig. 1 is a satisfactory approximation for the arc b-b'. Assume a curved beam of length L and constant I to be fixed at one end and to carry a constant moment M. The depth of the beam is L/20.

Change in slope at free end of beam will be $\Sigma\Delta\theta = \frac{ML}{EI}$

Substituting $\frac{fI}{c}$ for M $\qquad \Sigma\Delta\theta = \frac{fL}{Ec}$

The rotational deflection (bb') of the end segment is $\Delta\frac{SfL}{Ec}$

Using steel at f = 20, 000 psi and c = d/2 = L/40

$$bb' = \frac{20,000 \times 40}{30,000,000} \quad S = 0.027\,\Delta S$$

The corresponding value of the angle b ' bb' is 47 minutes. The average angular error in a beam where the moment runs from 0 to a maximum or from maximum negative to maximum positive values would be much smaller.

Stiff Beams. When the ratio d/R is comparatively large a modified moment of inertia must be used. Timoshenko and Young (Theory of Structures, page 423) show that $\Delta\theta = \frac{M\Delta S}{EI\left[1+\frac{4}{15}\left(\frac{d}{R}\right)^2\right]}$ for curved beams of rectangular section.

Values of the moment of inertia factor $\left[1+\frac{4}{15}\left(\frac{d}{R}\right)^2\right]$ for ratios of $\frac{d}{R}$ of $\frac{1}{5}$, $\frac{1}{10}$, $\frac{1}{20}$ are 1.107, 1.027 and 1.007, respectively.

TABLE I

Relative Deflections in Curved Beam of 12.75 Ft. Radius

Segment	M	$EI\frac{\Delta\theta\,\Delta s}{2}$	$EI\sum_{o}^{n-1}\Delta\theta$	$EI\Delta_S\sum_{o}^{n-1}\Delta\theta$	EI x Relative Deflection
1	2	3	4	5	6
1	100	200	0		200
2	100	200	200	400	600
3	100	200	400	800	1000
4	100	200	600	1200	1400
5	100	200	800	1600	1800
6	100	200	1000	2000	2200
7	100	200	1200	2400	2600
8	100	200	1400	2800	3000
9	100	200	1600	3200	3400
10	100	200	1800	3600	3800

TABLE II

Relative Deflections in Arch Due to 10 Kip Load at Point 6'

Segment	$\frac{\Delta_s}{I}$	M Kip Ft.	$\frac{M}{540} \cdot \frac{\Delta_s}{I}$	Adjust-ment	Adjusted $\Delta\theta$	Δ_s Inches	$\frac{\Delta_s}{2} \Delta\theta$	$\Sigma\Delta\theta$ o	$\Delta_s \Sigma\Delta\theta$ o	Relative Deflection
1	2	3	4	5	6	7	8	9	10	11=8+10
1	2.47	+70.41	+0.322	0.0001	+0.321	93.0	+14.93			+ 14.9
2	4.36	+41.88	+0.337	1	+0.336	87.8	+14.75	+0.321	+ 28.2	+ 43.0
3	6.89	+19.40	+0.247	1	+0.246	83.8	+10.32	+0.657	+ 55.1	+ 65.4
4	9.35	+ 2.01	+0.035	1	+0.034	80.5	+ 1.44	+0.903	+ 72.7	+ 74.1
5	10.71	-10.01	-0.1985	1	-0.1995	78.2	- 7.82	+0.937	+ 73.2	+ 65.4
6	11.53	-17.81	-0.380	1	-0.381	76.6	-14.60	+0.737	+ 56.5	+ 41.9
7	12.27	-22.14	-0.503	1	-0.502	75.2	-18.85	+0.356	+ 26.8	+ 8.0
8	12.98	-23.03	-0.553	2	-0.555	74.5	-20.7	-0.146	- 10.87	- 31.6
9	13.35	-20.89	-0.516	1	-0.517	74.0	-19.13	-0.701	- 52.0	- 71.1
10	13.54	-15.72	-0.394	1	-0.395	73.8	-14.56	-1.218	- 89.9	-104.5
10'	13.54	- 7.10	-0.1781	1	-0.179	73.8	- 6.60	-1.613	-119.0	-125.6
9'	13.35	+ 4.98	+0.1233	0	+0.123	74.0	+ 4.55	-1.792	-132.6	-128.0
8'	12.98	+20.08	+0.482	1	+0.481	74.5	+17.90	-1.629	-121.3	-103.4
7'	12.27	+38.21	+0.868	2	+0.866	75.2	+32.5	-1.148	- 86.4	- 53.9
6'	11.53	+59.79	+1.278	3	+1.275	76.6	+48.8	-0.282	- 21.6	+ 27.2
5'	10.71	+23.33	+0.462	1	+0.461	78.2	+18.02	+0.953	+ 74.5	+ 92.5
4'	9.35	- 8.81	-0.1525	0	-0.152	80.5	- 6.14	+1.414	+113.6	+107.5
3'	6.89	-35.77	-0.457	1	-0.458	83.8	-19.2	+1.262	+105.7	+ 86.5
2'	4.36	-57.55	-0.465	1	-0.466	87.8	-20.45	+0.804	+ 70.6	+ 50.1
1'	2.47	-73.27	-0.335	1	-0.336	93.0	-15.61	+0.338	+ 31.45	+ 15.8
0'		-78.20								

Columns 8, 10, 11 are in thousandths of an inch.

$E = 3 \times 10^6 \times 144 = 432 \times 10^6$ ft.^{4} $\quad L_T = 1.25I \quad \Delta\theta = \frac{M}{EI_t} \quad 10^6 \Delta\theta = \frac{M\Delta_s}{432 \times 1.25I} = \frac{M\Delta_s}{540I}$

TABLE III

Deflection and Influence Values for Two Span Continuous Beam

Segment	2M	$\sum_{o}^{n-1} 4M$	Relative Deflection	Vert. Dfln. Scaled	Influence Ordinate
1	2	3	4=2+3	5	6
a	18		18	85	0.128
b	14	36	50	200	0.302
c	10	64	74	342	0.516
d	6	84	90	498	0.752
e	2	96	98	663	1.000
a'			18	-49	0.074
b'			50	-65	0.098
c'			74	-56	0.084
d'			90	-33	0.045
e'			98	0	

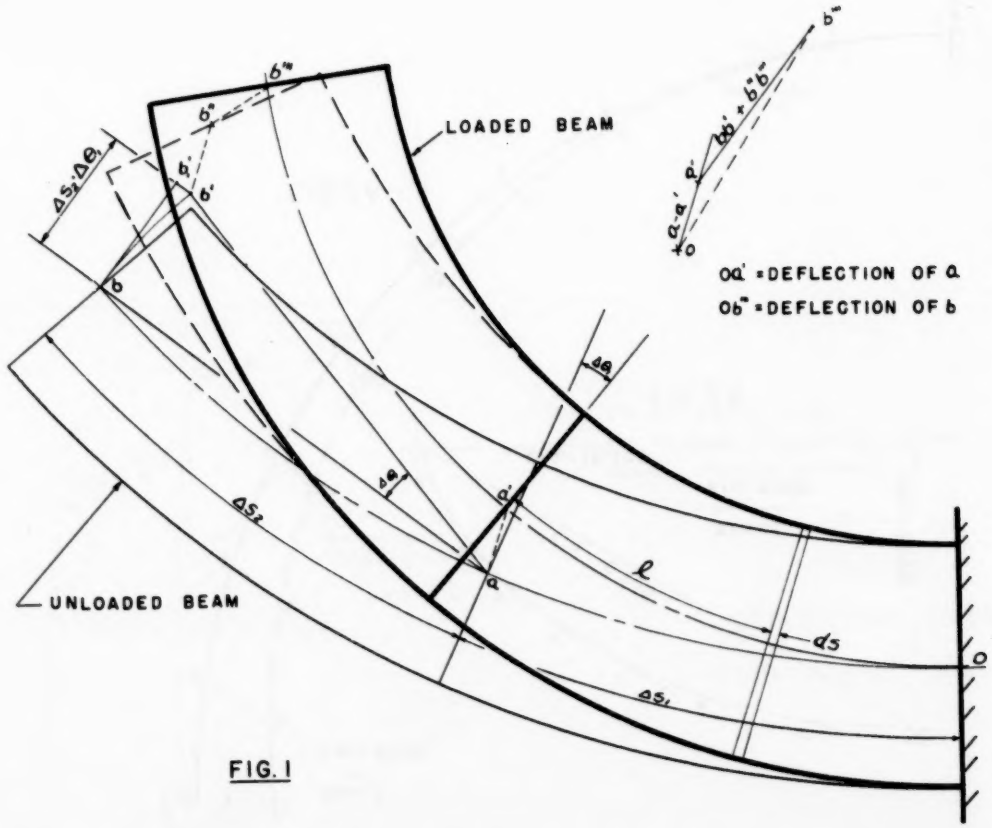

LOADED BEAM
UNLOADED BEAM
Oa' = DEFLECTION OF a
Ob''' = DEFLECTION OF b
ds
l

FIG. 1

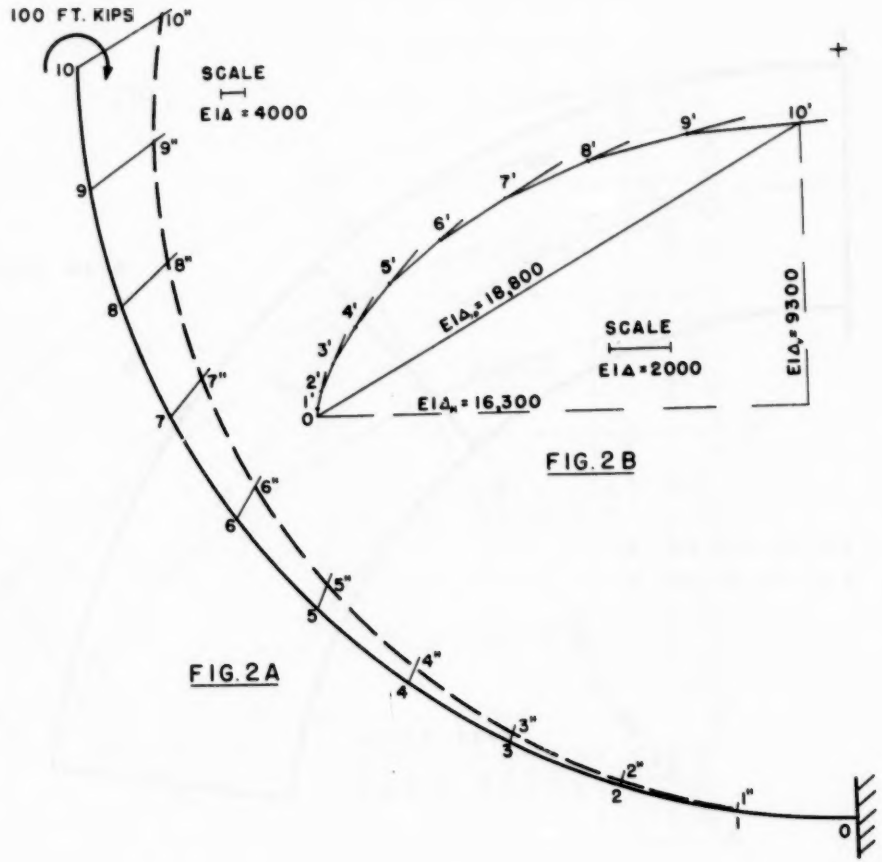

100 FT. KIPS
10"
10
SCALE
EIΔ = 4000
9"
9
8"
8
7"
7
6"
6
5"
5
4"
4
3"
3
2"
2
1"
1
0
+
10'
9'
8'
7'
6'
5'
4'
3'
2'
1'
0
EIΔ = 18,800
EIΔ = 16,300
SCALE
EIΔ = 2000
EIΔ = 9300

FIG. 2A

FIG. 2B

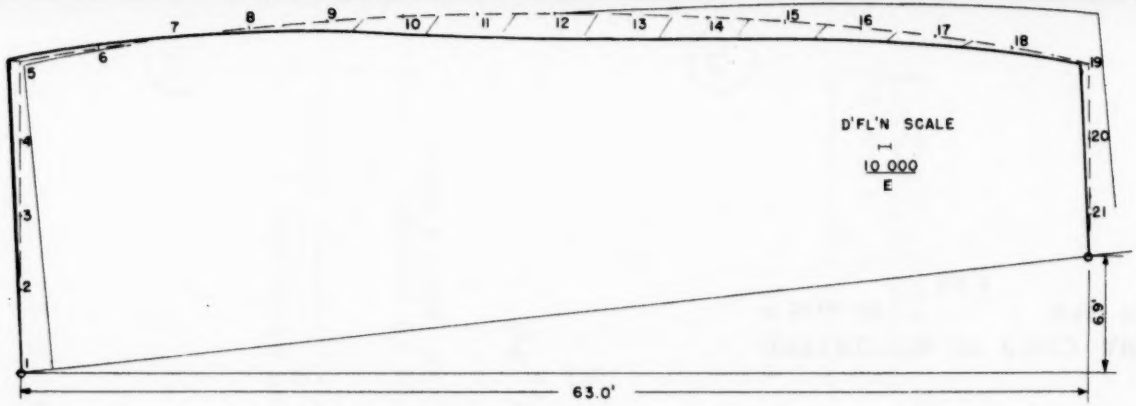


FIG. 3A

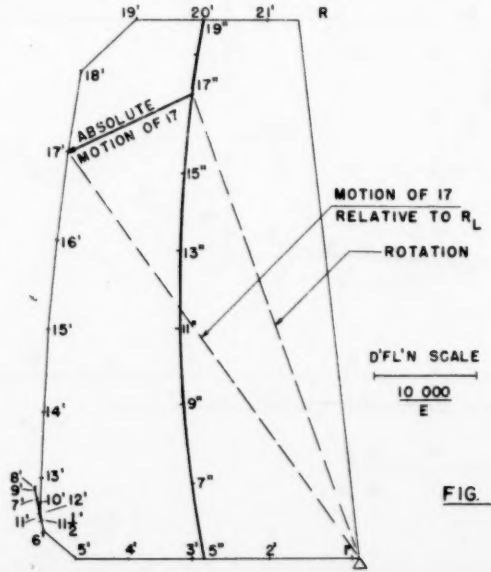


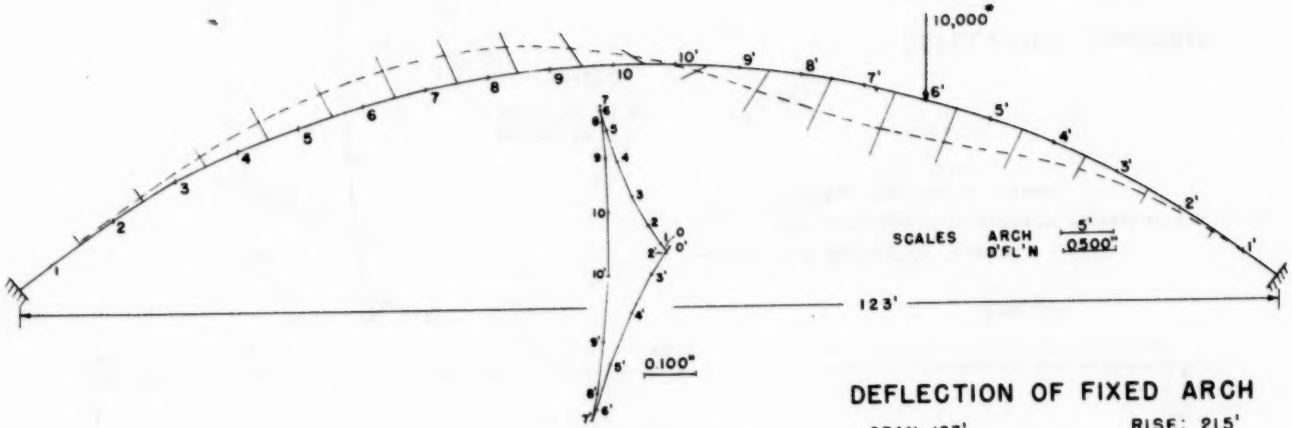
FIG. 3B

THE DEFLECTED FRAME

POS. OF FRAME WITH RESPECT TO TAN. AT STA. 11 1/2

ORIGINAL POSITION OF FRAME

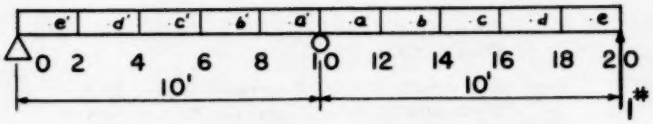
DEFLECTION DIAGRAM
RIGID FRAME BRIDGE



DEFLECTION OF FIXED ARCH

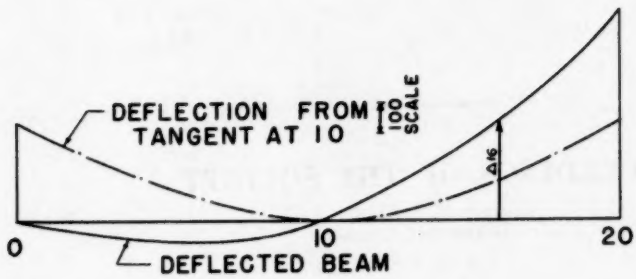
SPAN: 123' RISE: 21.5'

FIG. 4

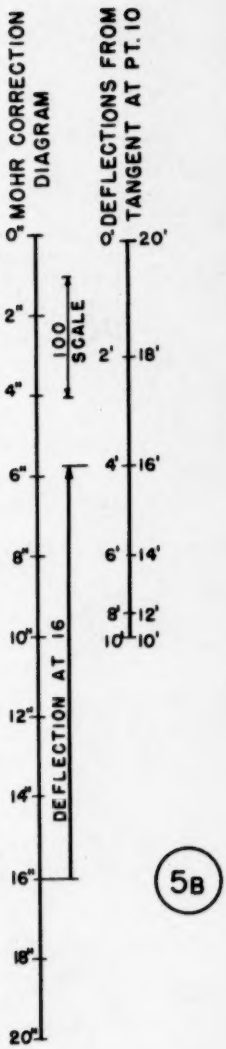


LOADED BEAM

5A



5C



5B

FIG. 5

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